

$$\begin{aligned}\int_0^1 \frac{x dx}{1+x^2} &= \frac{1}{2} \int_0^1 \frac{2x dx}{1+x^2} = \frac{1}{2} \int_0^1 \frac{dx^2}{1+x^2} = \frac{1}{2} \int_0^1 \frac{d(x^2+1)}{1+x^2} \\ &= \frac{1}{2} \ln |1+x^2| \Big|_0^1 = \frac{1}{2} (\ln |1+1| - \ln |1+0|) = \\ &= \frac{1}{2} (\ln 2 - \ln 1) = \frac{1}{2} \ln 2 = \ln \sqrt{2}\end{aligned}$$