$$\int_0^1 \frac{x\,dx}{1+x^2} = \frac{1}{2}\int_0^1 \frac{2x\,dx}{1+x^2} = \frac{1}{2}\int_0^1 \frac{dx^2}{1+x^2} = \frac{1}{2}\int_0^1 \frac{d(x^2+1)}{1+x^2} =$$

$$= \frac{1}{2}\ln|(1+x^2)|\Big|_0^1 = \frac{1}{2}\left(\ln|1+1| - \ln(1+0)\right) =$$

$$= \frac{1}{2}\left(\ln 2 - \ln 1\right) = \frac{1}{2}\ln 2 = \ln\sqrt{2}$$